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# An Approximate Solution for Homogeneous Boundary-Value Problems with Slowly-Varying Coefficient Matrices

S. M. SHAHRUZ\*

Berkeley Engineering Research Institute  
P.O. Box 9984, Berkeley, CA 94709, U.S.A.  
[shahrucz@robotics.eecs.berkeley.edu](mailto:shahrucz@robotics.eecs.berkeley.edu)

A. L. SCHWARTZ

Department of Electrical Engineering and Computer Sciences  
and the Electronics Research Laboratory  
University of California, Berkeley, CA 94720, U.S.A.

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**Abstract**—In this paper, an approximate closed-form solution for homogeneous boundary-value problems with slowly-varying coefficient matrices is obtained. The derivation of the approximate solution is based on the freezing technique, which is commonly used in analyzing the stability of slowly-varying initial-value problems as well as solving them. The error between the approximate and the exact solutions is given, and an upper bound on the norm of the error is obtained. This upper bound is proportional to the rate of change of the coefficient matrix of the boundary-value problem. The proposed approximate solution is obtained for a two-point boundary-value problem and is compared to its solution obtained numerically. Good agreement is observed between the approximate and the numerical solutions, when the rate of change of the coefficient matrix is small.

**Keywords**—Homogeneous boundary-value problem, Slowly-varying coefficients matrices, Approximate closed-form solution, Freezing technique, Error bound.

## 1. INTRODUCTION

Consider the linear boundary-value problem for ordinary differential equations of the form

$$y'(x) = A(x)y(x), \quad (1.1a)$$

where  $x \in (a, b) \subset \mathbb{R}$ , the unknown vector  $y(x) \in \mathbb{R}^n$ , and the matrix  $A(x) \in \mathbb{R}^{n \times n}$ , with the boundary conditions

$$B_a y(a) + B_b y(b) = \beta, \quad (1.1b)$$

where the matrices  $B_a, B_b \in \mathbb{R}^{n \times n}$ , and the vector  $\beta \in \mathbb{R}^n$ . The boundary-value problem represented by (1.1) frequently arises in engineering and physics.

The solution of the boundary-value problem (1.1) is obtained as follows (see, e.g., [1–3]):

- (i) Let  $Y(x) \in \mathbb{R}^{n \times n}$  be a matrix satisfying

$$Y'(x) = A(x)Y(x), \quad (1.2)$$

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\* Author to whom all correspondence should be addressed.

for all  $x \in [a, b]$ , and having linearly independent columns. (If the elements of the matrix  $A(\cdot)$  are continuous functions of  $x$  on  $[a, b]$ , then the columns of  $Y(x)$  are linearly independent for all  $x \in [a, b]$ .)

(ii) Define the constant matrix  $Q \in \mathbb{R}^{n \times n}$  by

$$Q := B_a Y(a) + B_b Y(b). \quad (1.3)$$

Suppose that the elements of the matrix  $A(\cdot)$  are continuous functions of  $x$  on  $[a, b]$ . The boundary-value problem (1.1) has a unique solution if and only if  $Q$  is a nonsingular matrix. The solution of (1.1) is given by

$$y(x) = Y(x) Q^{-1} \beta, \quad (1.4)$$

for all  $x \in [a, b]$ .

Computing the matrix  $Y(\cdot)$  in the solution (1.4) can be difficult. If  $A(\cdot)$  is a constant matrix, then  $Y(x) = e^{Ax}$  for all  $x \in [a, b]$ . If the elements of the matrix  $A(\cdot)$  are arbitrary functions of  $x$ , then it is not feasible to obtain  $Y(\cdot)$  for the linear boundary-value problem (1.1) in general. Thus, the solution (1.4) cannot be easily computed. When the elements of the matrix  $A(\cdot)$  are functions of  $x$ , one can solve (1.1) numerically (see, e.g., [1–5] and the references therein). However, contrary to initial-value problems, solving boundary-value problems numerically is difficult. Thus, any easy-to-compute approximate solution for boundary-value problems is welcome.

In this paper, we consider boundary-value problems represented by (1.1), and *assume* that:

- (A1) The elements of the matrix  $A(\cdot)$  are continuously differentiable and slowly-varying functions of  $x$  on  $[a, b]$ ;
- (A2) The problem (1.1) has a unique solution  $y(\cdot)$ .

For this class of problems, we obtain an approximate closed-form solution by the freezing technique. The freezing technique, which has not been used for boundary-value problems previously, is commonly used in:

- (i) establishing the stability of slowly-varying homogeneous initial-value problems (see, e.g., [6–8]);
- (ii) obtaining an approximate solution for slowly-varying inhomogeneous initial-value problems (see [9]).

The organization of the paper is as follows. In Section 2, we obtain our approximate solution for the boundary-value problem (1.1) by the freezing technique. In Section 3, we analyze the error in the approximate solution. In Section 4, we obtain our approximate solution for a two-point boundary-value problem and compare it to its solution obtained numerically.

## 2. AN APPROXIMATE SOLUTION BY THE FREEZING TECHNIQUE

We freeze the matrix  $A(\cdot)$  in (1.1) at a  $p \in [a, b]$ , and obtain the following boundary-value problem

$$y'(x; p) = A(p) y(x; p), \quad (2.1a)$$

where  $x \in (a, b) \subset \mathbb{R}$ , the unknown vector  $y(x; p) \in \mathbb{R}^n$ , and the *constant* matrix  $A(p) \in \mathbb{R}^{n \times n}$ , with the boundary conditions

$$B_a y(a; p) + B_b y(b; p) = \beta. \quad (2.1b)$$

We call (2.1) the *frozen* boundary-value problem corresponding to (1.1) at  $p$ .

The matrix satisfying  $Y'(x; p) = A(p) Y(x; p)$ , and having linearly independent columns is

$$Y(x; p) = e^{A(p)x}, \quad (2.2)$$

for all  $x \in [a, b]$ . We define

$$Q(p) := B_a e^{A(p)a} + B_b e^{A(p)b}. \quad (2.3)$$

Suppose that  $Q(p)$  is nonsingular for all  $p \in [a, b]$ . By (1.4), the solution of (2.1) for a fixed  $p \in [a, b]$  is

$$y(x; p) = e^{A(p)x} \left[ B_a e^{A(p)a} + B_b e^{A(p)b} \right]^{-1} \beta, \quad (2.4)$$

for all  $x \in [a, b]$ .

We now formally replace  $p$  by  $x$  in (2.4), and denote the resulting vector-valued function of  $x$  by  $\tilde{y}(\cdot)$ . We have

$$\tilde{y}(x) = e^{A(x)x} \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \beta, \quad (2.5)$$

for all  $x \in [a, b]$ . We choose  $\tilde{y}(\cdot)$  as an approximate solution for the boundary-value problem (1.1). Next, we show that  $\tilde{y}(\cdot)$  is a reasonably accurate approximate solution for (1.1), when the rates of change of the elements of the matrix  $A(\cdot)$  with respect to  $x$  are small.

### 3. APPROXIMATION ERROR

We evaluate the accuracy of the approximate solution  $\tilde{y}(\cdot)$  in (2.5) by obtaining the error between  $y(\cdot)$ , the exact solution of (1.1), and  $\tilde{y}(\cdot)$ . We define the error by

$$\varepsilon(x) := y(x) - \tilde{y}(x), \quad (3.1)$$

for all  $x \in [a, b]$ . We now provide a formula for  $\varepsilon(\cdot)$ .

LEMMA 3.1. *The error  $\varepsilon(\cdot)$  in (3.1) satisfies*

$$\begin{aligned} \varepsilon(x) = & -e^{A(x)x} \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} B_b \int_a^b e^{A(x)(b-\xi)} (A(\xi) - A(x)) y(\xi) d\xi \\ & + \int_a^x e^{A(x)(x-\xi)} (A(\xi) - A(x)) y(\xi) d\xi, \end{aligned} \quad (3.2)$$

for all  $x \in [a, b]$ .

PROOF. We show that (3.1) and (3.2) are equivalent as follows. Using (1.1a), we have

$$\int_a^x e^{A(x)(x-\xi)} (A(\xi) - A(x)) y(\xi) d\xi = e^{A(x)x} \int_a^x e^{-A(x)\xi} (y'(\xi) - A(x)y(\xi)) d\xi, \quad (3.3)$$

for all  $x \in [a, b]$ . Integrating by parts, we obtain

$$\int_a^x e^{-A(x)\xi} y'(\xi) d\xi = e^{-A(x)x} y(x) - e^{-A(x)a} y(a) + \int_a^x e^{-A(x)\xi} A(x) y(\xi) d\xi, \quad (3.4)$$

for all  $x \in [a, b]$ . Substituting (3.4) into (3.3), we obtain

$$\int_a^x e^{A(x)(x-\xi)} (A(\xi) - A(x)) y(\xi) d\xi = y(x) - e^{A(x)(x-a)} y(a), \quad (3.5)$$

for all  $x \in [a, b]$ . Similarly, we obtain

$$\int_a^b e^{A(x)(b-\xi)} (A(\xi) - A(x)) y(\xi) d\xi = y(b) - e^{A(x)(b-a)} y(a), \quad (3.6)$$

for all  $x \in [a, b]$ . Substituting (3.5) and (3.6) into (3.2), we obtain

$$\begin{aligned} \varepsilon(x) = & -e^{A(x)x} \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \left[ B_b y(b) - B_b e^{A(x)(b-a)} y(a) \right] \\ & + y(x) - e^{A(x)(x-a)} y(a), \end{aligned} \quad (3.7)$$

for all  $x \in [a, b]$ . We rewrite  $\varepsilon(\cdot)$  in (3.7) as

$$\begin{aligned} \varepsilon(x) = & e^{A(x)x} \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \left[ \beta - B_b y(b) + B_b e^{A(x)(b-a)} y(a) \right] \\ & - e^{A(x)x} \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \beta + y(x) - e^{A(x)(x-a)} y(a), \end{aligned} \quad (3.8)$$

for all  $x \in [a, b]$ , where we recall that  $\beta = B_a y(a) + B_b y(b)$ . Thus, using (2.5), we can rewrite (3.8) as

$$\begin{aligned} \varepsilon(x) = & y(x) - \tilde{y}(x) \\ & + e^{A(x)x} \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \left[ B_a y(a) + B_b e^{A(x)(b-a)} y(a) \right] \\ & - e^{A(x)(x-a)} y(a), \end{aligned} \quad (3.9)$$

for all  $x \in [a, b]$ . We have

$$B_a y(a) + B_b e^{A(x)(b-a)} y(a) = \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right] e^{-A(x)a} y(a), \quad (3.10)$$

for all  $x \in [a, b]$ . Substituting (3.10) into (3.9), we obtain  $\varepsilon(x) = y(x) - \tilde{y}(x)$  for all  $x \in [a, b]$ . That is,  $\varepsilon(\cdot)$  in (3.1) satisfies (3.2).  $\blacksquare$

Having the error  $\varepsilon(\cdot)$  in (3.2), we determine an upper bound on the norm of the error. We show that the error bound is proportional to the rates of change of the elements of the matrix  $A(\cdot)$  with respect to  $x$ . In the following, we make use of:

- (i) The  $L_\infty$ -norm of vectors  $\nu(x) = [\nu_1(x) \ \nu_2(x) \ \dots \ \nu_n(x)]^\top \in \mathbb{R}^n$  with continuous elements, defined over  $[a, b]$  by

$$\|\nu\|_\infty := \max_{1 \leq i \leq n} \max_{x \in [a, b]} |\nu_i(x)|. \quad (3.11)$$

- (ii) The  $L_\infty$ -induced norm of matrices  $M(x) = [m_{ij}(x)] \in \mathbb{R}^{n \times n}$  with continuous elements, defined at an  $x \in [a, b]$  and over  $[a, b]$ , respectively, by

$$\|M(x)\|_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^n |m_{ij}(x)|, \quad (3.12a)$$

$$\|M\|_\infty := \max_{1 \leq i \leq n} \max_{x \in [a, b]} \sum_{j=1}^n |m_{ij}(x)|. \quad (3.12b)$$

- (iii) The  $L_2$ -induced norm of the inverse of square nonsingular matrices  $M(x) \in \mathbb{R}^{n \times n}$  with continuous elements, given at an  $x \in [a, b]$  by

$$\|M^{-1}(x)\|_2 = \frac{1}{\sigma_{\min}(M(x))}, \quad (3.13)$$

where  $\sigma_{\min}(M(x))$  denotes the smallest singular value of the matrix  $M(x)$ .

- (iv) The relation between the norms in (3.12a) and (3.13)

$$\|M^{-1}(x)\|_\infty \leq \sqrt{n} \|M^{-1}(x)\|_2, \quad (3.14)$$

for all  $x \in [a, b]$ , which is due to the equivalence of these norms (see, e.g., [10, p. 314]).

With these preliminaries, we give an upper bound on  $\varepsilon(\cdot)$ :

**THEOREM 3.2.** *The norm of the error  $\varepsilon(\cdot)$  in (3.1) (equivalently (3.2)) satisfies*

$$\|\varepsilon\|_\infty \leq \frac{e^{\|A\|_\infty(b-a)}(b-a)^2}{2} \left( \frac{\sqrt{n} e^{\|A\|_\infty b} \|B_b\|_\infty}{\sigma} + 1 \right) \|A'\|_\infty \|y\|_\infty, \quad (3.15)$$

where

$$\sigma := \min_{x \in [a, b]} \sigma_{\min} \left( B_a e^{A(x)a} + B_b e^{A(x)b} \right). \quad (3.16)$$

PROOF. From (3.2) we have

$$\begin{aligned} \|\varepsilon\|_{\infty} &\leq \left\| e^{A(x)x} \right\|_{\infty} \left\| \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \right\|_{\infty} \|B_b\|_{\infty} \\ &\quad \times \int_a^b \left\| e^{A(x)(b-\xi)} \right\|_{\infty} \|A(\xi) - A(x)\|_{\infty} \|y\|_{\infty} d\xi \\ &\quad + \int_a^x \left\| e^{A(x)(x-\xi)} \right\|_{\infty} \|A(\xi) - A(x)\|_{\infty} \|y\|_{\infty} d\xi, \end{aligned} \quad (3.17)$$

for all  $x \in [a, b]$ .

We have

$$\left\| e^{A(x)x} \right\|_{\infty} \leq e^{\|A(x)x\|_{\infty}} \leq e^{\|A\|_{\infty} b}, \quad (3.18)$$

and

$$\left\| e^{A(x)(b-\xi)} \right\|_{\infty} \leq e^{\|A(x)(b-\xi)\|_{\infty}} \leq e^{\|A\|_{\infty}(b-a)}, \quad (3.19a)$$

$$\left\| e^{A(x)(x-\xi)} \right\|_{\infty} \leq e^{\|A(x)(x-\xi)\|_{\infty}} \leq e^{\|A\|_{\infty}(b-a)}, \quad (3.19b)$$

for all  $x$  and  $\xi$  in  $[a, b]$ .

By (3.14), (3.13), and (3.16), we have

$$\begin{aligned} \left\| \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \right\|_{\infty} &\leq \sqrt{n} \left\| \left[ B_a e^{A(x)a} + B_b e^{A(x)b} \right]^{-1} \right\|_2 \\ &= \frac{\sqrt{n}}{\sigma_{\min}(B_a e^{A(x)a} + B_b e^{A(x)b})} \leq \frac{\sqrt{n}}{\sigma}, \end{aligned} \quad (3.20)$$

for all  $x \in [a, b]$ .

Furthermore, we have

$$A(\xi) - A(x) = \int_x^{\xi} A'(\eta) d\eta, \quad (3.21)$$

for all  $\eta \in [x, \xi]$ . Thus,

$$\|A(\xi) - A(x)\|_{\infty} \leq \|A'\|_{\infty} |\xi - x|, \quad (3.22)$$

for all  $x \in \xi$  and  $x$  and  $\xi$  in  $[a, b]$ .

Substituting (3.18)–(3.20) and (3.22) into (3.17) and carrying out the integrations, we then obtain (3.15).  $\blacksquare$

The upper bound on  $\|\varepsilon\|_{\infty}$  in (3.15) is proportional to  $\|A'\|_{\infty}$  and  $\|y\|_{\infty}$ . By (A2), the boundary-value problem (1.1) has a unique solution for  $y(\cdot)$  on  $[a, b]$ ; thus  $\|y\|_{\infty} < \infty$ . If  $\|A'\|_{\infty}$  is sufficiently small, i.e., if the rates of change of the elements of the matrix  $A(\cdot)$  with respect to  $x$  are sufficiently small, then  $\|\varepsilon\|_{\infty}$  is small.

#### 4. EXAMPLE

In this section, we give an example to illustrate the application of the theory developed in the previous sections. We obtain the approximate solution in (2.5) for a two-point boundary-value problem and compare it to its solution obtained numerically.

We consider the scalar two-point boundary-value problem

$$z''(x) + 2d(x) z'(x) + k(x) z(x) = 0, \quad (4.1a)$$

where  $x \in (0, x_1) \subset \mathbb{R}$ ,  $z(x) \in \mathbb{R}$ , the coefficients  $d(x)$ ,  $k(x) \in \mathbb{R}$ , and the boundary conditions are

$$z(0) = z_0 \in \mathbb{R}, \quad z(x_1) = z_1 \in \mathbb{R}. \quad (4.1b)$$

We let  $y_1(x) := z(x)$  and  $y_2(x) := z'(x)$ , and  $y(x) := [y_1(x) \ y_2(x)]^\top$  for all  $x \in [0, x_1]$ . Then the boundary-value problem (4.1) can be written as

$$y'(x) = A(x) y(x), \quad (4.2)$$

for all  $x \in (0, x_1)$ , where

$$A(x) = \begin{bmatrix} 0 & 1 \\ -k(x) & -2d(x) \end{bmatrix}, \quad (4.3)$$

with the boundary conditions  $y_1(0) = z_0$  and  $y_1(x_1) = z_1$ . We rewrite the boundary conditions as (1.1b) with

$$B_a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}. \quad (4.4)$$

The boundary-value problem (4.1) (equivalently (4.2)–(4.4)) cannot be solved analytically for  $z(\cdot)$ , when  $d(\cdot)$  and  $k(\cdot)$  are arbitrary functions of  $x$ . It is, however, possible to obtain an approximate solution for (4.1), when the coefficients  $d(\cdot)$  and  $k(\cdot)$  are slowly-varying functions of  $x$ . We obtain this approximate solution by the freezing technique presented in Section 2.

The frozen boundary-value problem corresponding to (4.1) at  $x = p$  is

$$y'(x; p) = A(p) y(x; p), \quad (4.5)$$

for all  $x \in (0, x_1)$ , where

$$A(p) = \begin{bmatrix} 0 & 1 \\ -k(p) & -2d(p) \end{bmatrix}, \quad (4.6)$$

with the boundary conditions  $y_1(0; p) = z_0$  and  $y_1(x_1; p) = z_1$ .

For the fixed matrix  $A(p)$  in (4.6), we obtain

$$Y(x; p) = e^{A(p)x} = \begin{bmatrix} \alpha_0(x; p) & \alpha_1(x; p) \\ -k(p)\alpha_1(x; p) & \alpha_0(x; p) - 2d(p)\alpha_1(x; p) \end{bmatrix}, \quad (4.7)$$

where

$$\alpha_0(x; p) := \frac{\lambda_2(p) e^{\lambda_1(p)x} - \lambda_1(p) e^{\lambda_2(p)x}}{\lambda_2(p) - \lambda_1(p)}, \quad (4.8a)$$

$$\alpha_1(x; p) := \frac{e^{\lambda_2(p)x} - e^{\lambda_1(p)x}}{\lambda_2(p) - \lambda_1(p)}, \quad (4.8b)$$

for all  $x$  and  $p$  in  $[0, x_1]$ , with

$$\lambda_1(p) := -d(p) - (d^2(p) - k(p))^{1/2}, \quad (4.9a)$$

$$\lambda_2(p) := -d(p) + (d^2(p) - k(p))^{1/2}, \quad (4.9b)$$

Thus, from (2.3) we have

$$Q(p) = \begin{bmatrix} 1 & 0 \\ \alpha_0(x_1; p) & \alpha_1(x_1; p) \end{bmatrix}. \quad (4.10)$$

We substitute (4.7) and (4.10) into (1.4) and obtain  $z(x; p) := y_1(x; p)$ . Then, we formally replace  $p$  by  $x$  in  $z(x; p)$ , and denote the resulting function of  $x$  by  $\tilde{z}(\cdot)$ . We have

$$\tilde{z}(x) = \frac{e^{\lambda_1(x)x + \lambda_2(x)x_1} - e^{\lambda_2(x)x + \lambda_1(x)x_1}}{e^{\lambda_2(x)x_1} - e^{\lambda_1(x)x_1}} z_0 + \frac{e^{\lambda_2(x)x} - e^{\lambda_1(x)x}}{e^{\lambda_2(x)x_1} - e^{\lambda_1(x)x_1}} z_1, \quad (4.11)$$

where

$$\lambda_1(x) := -d(x) - (d^2(x) - k(x))^{1/2}, \quad (4.12a)$$

$$\lambda_2(x) := -d(x) + (d^2(x) - k(x))^{1/2}, \quad (4.12b)$$

for all  $x \in [0, x_1]$ .

We choose  $\tilde{z}(\cdot)$  in (4.11) as an approximate solution of (4.1). According to our analysis, this approximate solution is reasonably accurate, when the rates of change of the elements of the matrix  $A(\cdot)$ , namely,  $d(\cdot)$  and  $k(\cdot)$ , with respect to  $x$  are small. We confirm this fact by an example.

Consider the two-point boundary-value problem (4.1) defined on the interval  $(0, 2)$  with the coefficients

$$d(x) = 1.5 + 0.5 \sin \omega x, \quad (4.13a)$$

$$k(x) = e^{-\alpha x}, \quad (4.13b)$$

where  $\omega > 0$  and  $\alpha > 0$  are constant numbers, and the boundary conditions are  $z_0 = z(0) = 1$  and  $z_1 = z(2) = 2$ . We used (4.11) to obtain an approximate solution for (4.1) and also solved (4.1) numerically by the shooting technique (see, e.g., [1-3]) for different values of  $\omega$  and  $\alpha$ . The approximate and the numerical solutions, and the error between them are depicted in:

- (i) Figure 1, for  $\omega = 0.01$  and  $\alpha = 0.01$ ;
- (ii) Figure 2, for  $\omega = 0.1$  and  $\alpha = 0.1$ .

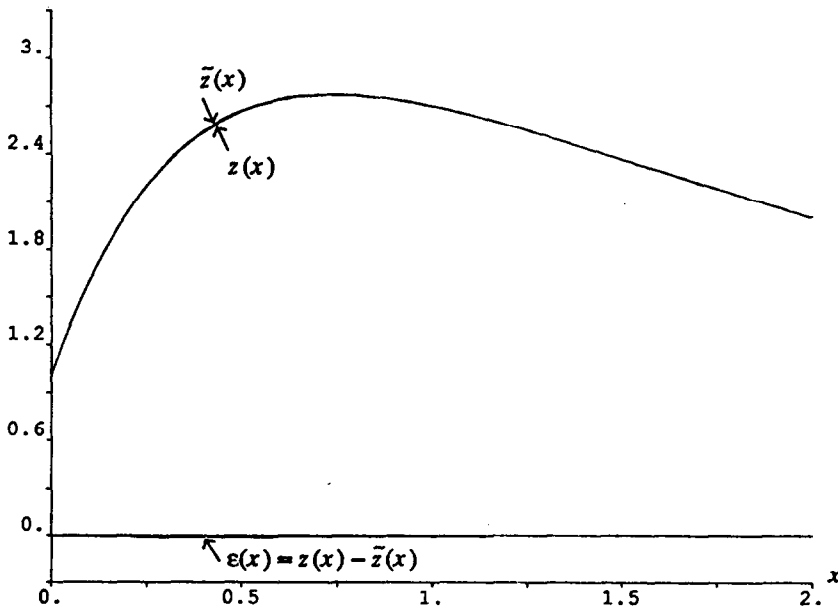


Figure 1. The numerical solution  $z$ , and the approximate solution,  $\tilde{z}$ , of the boundary-value problem  $z''(x) + 2(1.5 + 0.5 \sin \omega x)z'(x) + e^{-\alpha x}z(x) = 0$ , with the boundary conditions  $z(0) = 1$  and  $z(2) = 2$ , when  $\omega = 0.01$  and  $\alpha = 0.01$ . The error is denoted by  $\epsilon$ . The error is small because the rates of change of the coefficients of  $z'$  and  $z$  in the equation are small.

It is evident that the approximate and the numerical solutions of (4.1) in Figure 1 are closer to each other than those in Figure 2. This is due to the fact that for  $\omega$  and  $\alpha$  in (i) the rates of change of  $d(\cdot)$  and  $k(\cdot)$  are smaller.

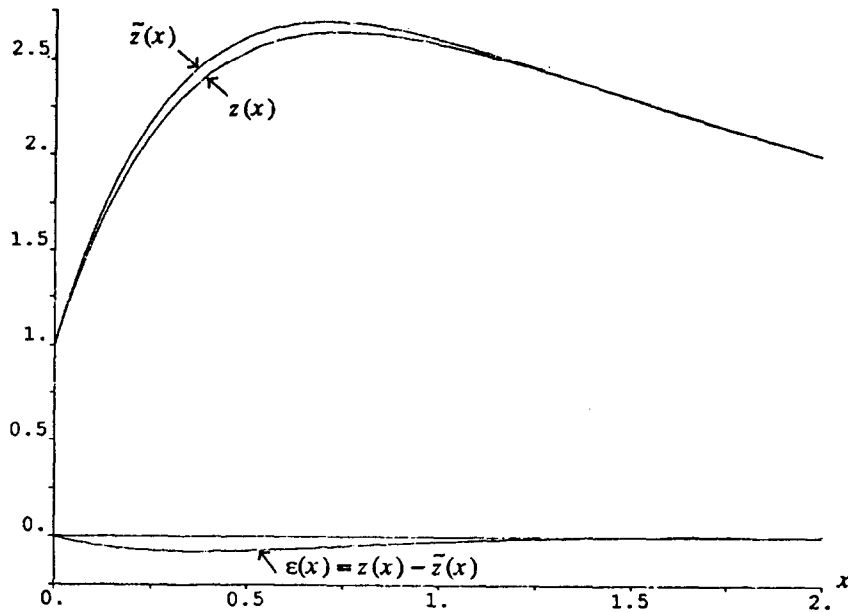


Figure 2. The numerical solution,  $z$ , and the approximate solution,  $\bar{z}$ , of the boundary-value problem  $z''(x) + 2(1.5 + 0.5 \sin \omega x)z'(x) + e^{-\alpha x}z(x) = 0$ , with the boundary-value conditions  $z(0) = 1$  and  $z(2) = 2$ , when  $\omega = 0.1$  and  $\alpha = 0.1$ . The error is denoted by  $\varepsilon$ .

## 5. CONCLUSIONS

In this paper, we obtained an approximate solution for homogeneous boundary-value problems with slowly-varying coefficient matrices. We obtained the approximate solution by the freezing technique. This technique is straightforward and is as follows:

- (i) freeze the coefficient matrix of the boundary-value problem at  $x = p$ ;
- (ii) solve the frozen boundary-value problem;
- (iii) replace the variable  $p$  by  $x$  in the solution of the frozen problem to obtain an approximate solution for the slowly-varying problem.

We obtained an upper bound on the norm of the error between the approximate and the exact solutions of the problem. This upper bound is proportional to the rate of change of the elements of the coefficient matrix of the boundary-value problem; thus, the proposed approximate solution is reasonably accurate, when the elements of this matrix vary slowly.

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